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Korovkin-type theorem and application

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Abstract

Let (L_n) be a sequence of positive linear operators on $C[0, 1]$, satisfying that $(L_n(e_i))$ converge in $C[0, 1]$ (not necessarily to e_i) for $i = 0, 1, 2$, where $e_i(x) = x^i$. We prove that the conditions that (L_n) is monotonicity-preserving, convexity-preserving and variation diminishing do not suffice to insure the convergence of $(L_n(f))$ for all $f \in C[0, 1]$. We obtain the Korovkin-type theorem and give quantitative results for the approximation properties of the q-Bernstein operators $B_{n,q}$ as an application.

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Keywords: Korovkin-type theorem; q-Bernstein operators

1. Introduction

Let $(L_n)_{n\geqslant 1}$ be a sequence of positive linear operators on $C[0, 1]$. We say that L_n is monotonicity-preserving (or convexity-preserving) if $L_n(f)$ is increasing (or convex) for an increasing (or convex) function *f*. For any real sequence *a*, finite or infinite, we denote by $S^-(a)$ the number of strict sign changes in *a*. For $f \in C[0, 1]$, we define $S^-(f)$ to be the number of sign changes of *f*, that is

$$S^-(f) = \sup\ S^-(f(x_0), \ldots, f(x_m)),$$

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where the supremum is taken over all increasing sequences $0 \leqslant x_0 < \cdots < x_m \leqslant 1$ for all positive integers m. We say that L_n is variation diminishing if for all functions $f \in C[0,1]$, we have

$$S^-(L_n f) \leqslant S^-(f).$$

From the well-known Korovkin theorem, we have the convergence $L_n(f) \to f$ in the uniform norm for all $f \in C[0,1]$, if it holds for the test functions $e_i(x) = x^i,\ i = 0, 1, 2$ (see [1]). However, in studying the approximating properties of the q-Bernstein operators (see Section 3), we encounter the following problem: the sequences $(L_n(e_i))_{n \geqslant 1}$ converge in $C[0,1]$ but not necessarily to e_i for $i = 0, 1, 2$. It is natural to ask whether the convergence of $(L_n(e_i))_{n \geqslant 1}$, $i = 0, 1, 2$ implies that there exists an operator L_∞ on $C[0,1]$ such that $\|L_n(f) - L_\infty(f)\| \to 0$ for each $f \in C[0,1]$, here $\|\cdot\|$ represents the uniform norm. In general, the answer is negative. In order to insure the existence of L_∞, we must add some conditions. Which of the following conditions can guarantee this?

Condition A: (L_n) is monotonicity-preserving and convexity-preserving.

Condition B: (L_n) is variation diminishing.

Condition C: $(L_n(f, x))_{n \geqslant 1}$ is non-increasing for any convex function f and any $x \in [0, 1]$.

We assert that Conditions A and B do not suffice to insure the convergence of $(L_n(f))$ for all $f \in C[0,1]$. We shall give examples. But if we assume (L_n) satisfies Condition C, we can show the existence of L_∞. This is our Korovkin-type theorem. Also, the rate of approximation $|L_n(f, x) - L_\infty(f, x)|$ can be estimated by the smoothness of f and the quantity $|L_n(e_2, x) - L_\infty(e_2, x)|$. These statements are proved in Section 2. In Section 3, as an application of the above Korovkin-type theorem, we give quantitative results for the approximation properties of the q-Bernstein operators. Note that the q-Bernstein operators satisfy Conditions A–C (see Section 3).

Now we formulate the main results of the paper. For $f \in C[0,1],\ t > 0$, the second modulus of smoothness of f is defined by

$$\omega_2(f, t) = \sup_{0 < h \leqslant t}\ \sup_{x \in [0, 1-2h]} |f(x + 2h) - 2f(x + h) + f(x)|.$$

Theorem 1. *There exists a sequence $(L_n)_{n \geqslant 1}$ of positive linear operators on $C[0,1]$ such that*

(a) *the sequences $(L_n(e_i))$ converge in $C[0,1]$ for $i = 0, 1, 2$, where $e_i(x) = x^i$,*
(b) *(L_n) satisfies Conditions* A *and* B*, and*
(c) *there exists a function $f \in C[0,1]$ such that $(L_n(f))$ does not converge in $C[0,1]$.*

Theorem 2. *Let the sequence (L_n) of positive linear operators on $C[0,1]$ satisfy the following conditions*:

(d) *the sequence $(L_n(e_2))$ converges to a function $L_\infty(e_2)$ in $C[0,1]$,*
(e) *(L_n) satisfies Condition* C.

Then there exists an operator L_∞ *on* $C[0,1]$ *such that* $\|L_n(f) - L_\infty(f)\| \to 0$ *for every* $f \in C[0,1]$. *Furthermore,*

$$|L_n(f,x) - L_\infty(f,x)| \leqslant c\,\omega_2(f, \sqrt{\lambda_n(x)}), \tag{1.1}$$

where $\lambda_n(x) = |L_n(e_2,x) - L_\infty(e_2,x)|$, *c is a constant depending only on* $\|L_1(e_0)\|$.

2. Proofs of Theorems 1 and 2

Proof of Theorem 1. First, we construct linear operators $L(\cdot,\xi)$ on $C[0,1]$ with $\xi \in [1/3, 2/3]$. For $f \in C[0,1]$, let

$$L(f,\xi,x) = f(0)a(x,\xi) + f(\xi)b(x,\xi) + f(1)c(x,\xi),$$

where $a(x,\xi),\ b(x,\xi),\ c(x,\xi)$ satisfy the system of equations:

$$\begin{cases} a(x,\xi) + b(x,\xi) + c(x,\xi) = 1, \\ b(x,\xi)\cdot\xi + c(x,\xi) = x, \\ b(x,\xi)\cdot\xi^2 + c(x,\xi) = g(x), \end{cases} \qquad g(x) = \begin{cases} 2x/3, & 0 \leqslant x \leqslant 1/2, \\ 4x/3 - 1/3, & 1/2 < x \leqslant 1. \end{cases}$$

Solving the system, we get

$$a(x,\xi) = \frac{\xi + g(x) - (1+\xi)x}{\xi}, \quad b(x,\xi) = \frac{x - g(x)}{\xi - \xi^2}, \quad c(x,\xi) = \frac{g(x) - \xi x}{1-\xi},$$

For $f \in C[0,1],\ \xi \in [1/3, 2/3]$, by the definitions of $L(\cdot,\xi)$ and $g(x)$, we know that

$$L(e_0,\xi) = e_0, \quad L(e_1,\xi) = e_1, \quad L(e_2,\xi) = g, \tag{2.1}$$

$$L(f,\xi,0) = f(0), \quad L(f,\xi,1) = f(1) \tag{2.2}$$

and

$$\xi x \leqslant 2x/3 \leqslant g(x) \leqslant x;$$

$$\xi + g(x) - (1+\xi)x = \begin{cases} (\xi(1-x)/x - 1/3)x, & 0 \leqslant x \leqslant 1/2 \\ (\xi - 1/3)(1-x), & 1/2 < x \leqslant 1 \end{cases} \geqslant 0.$$

Hence $a(x,\xi),\ b(x,\xi),\ c(x,\xi) \geqslant 0$ and therefore the operators $L(\cdot,\xi)$ are positive linear operators. From (2.1), we know that the operators $L(\cdot,\xi)$ reproduce linear functions. Furthermore, $L(\cdot,\xi)$ are monotonicity-preserving and convexity-preserving. In fact, if *f* is increasing on [0, 1], then

$$3\xi(1-\xi)\frac{d(L(f,\xi,x))}{dx} = \begin{cases} (f(\xi) - f(0)) + \xi(2-3\xi)(f(1) - f(0)), & 0 \leqslant x < 1/2 \\ (f(1) - f(\xi)) + (3\xi - 1)(1-\xi)(f(1) - f(0)), & 1/2 < x \leqslant 1 \end{cases} \geqslant 0.$$

So $L(f, \xi, x)$ are also increasing on [0, 1]. If f is convex on [0, 1], then for $x \in [0, 1/2)$, $y \in (1/2, 1]$, we have

$$3\xi(1-\xi)\left(\frac{d(L(f,\xi,y))}{dy}-\frac{d(L(f,\xi,x))}{dx}\right)=2((1-\xi)f(0)+\xi f(1)-f(\xi))\geqslant 0.$$

Hence $L(f, \xi, x)$ are also convex on [0, 1].

Next we show that the operators $L(\cdot, \xi)$ are variation diminishing. For $f \in C[0, 1]$, $L(f, \xi)$ are piecewise linear and continuous, by (2.2) we have

$$\begin{aligned} S^-(L(f,\xi)) &= S^-(L(f,\xi,0), L(f,\xi,1/2), L(f,\xi,1)) \\ &= S^-(f(0), L(f,\xi,1/2), f(1)) \leqslant 2. \end{aligned} \tag{2.3}$$

If $S^-(f) = 0$ or $\geqslant 2$, from the positivity of $L(\cdot, \xi)$ and (2.3), we get

$$S^-(L(f,\xi)) \leqslant S^-(f). \tag{2.4}$$

Suppose that (2.4) fails for some $f \in C[0, 1]$. Then $S^-(f) = 1$ and $S^-(L(f, \xi)) = S^-(f(0), L(f, \xi, 1/2), f(1)) = 2$. Hence $f(0) \cdot f(1) > 0$. For arbitrary $\xi \in [1/3, 2/3]$, $S^-(f(0), f(\xi), f(1)) \leqslant S^-(f) = 1$, thus we obtain that $f(\xi) \cdot f(0) \geqslant 0$. Since $S^-(f(0), f(\xi), f(1)) = 0$, we have $S^-(L(f, \xi)) = 0$. This leads to a contradiction. So (2.4) holds. The operators $L(\cdot, \xi)$ are variation diminishing.

Now we construct the sequence of linear positive operators (L_n) on $C[0, 1]$. Let $\xi(n) = (1 + |\sin n|)/3$. Then $\xi(n) \in [1/3, 2/3]$ and $\{\xi(n)\}_{n \geqslant 1}$ diverges. Let $L_n(f) = L(f, \xi(n))$. Then (L_n) satisfies (a) and (b). Let the continuous function f be such that $f(0) = 1$ and $f(x) = 0$ for $x \in [1/4, 1]$. Then $(L_n(f)) = \left(1 - x + \frac{g(x)-x}{\xi(n)}\right)$ diverges in $C[0, 1]$. The proof of Theorem 1 is complete. □

Proof of Theorem 2. First we show the existence of the operator L_∞. Let the sequence (L_n) of positive operators satisfy (d) and (e). Then $L_n(l) = L_m(l)$ for any linear function l, and the uniform norm $\sup_{n \geqslant 1} \|L_n\|$ of (L_n) satisfies

$$\sup_{n \geqslant 1} \|L_n\| \leqslant \sup_{n \geqslant 1} \|L_n(e_0)\| = \|L_1(e_0)\| < +\infty.$$

By the well-known Banach–Steinhaus theorem (see [1]), we know it suffices to prove the convergence of the sequence $(L_n(f))$ in $C[0, 1]$ for each $f \in C^2[0, 1]$, since the space $C^2[0, 1]$ is dense in $C[0, 1]$, where $C^2[0, 1]$ denotes the space of twice continuously differentiable functions on [0, 1].

For any $f \in C^2[0, 1]$, we know that the functions $g_1(x) = \frac{\|f''\|}{2}x^2 - f(x)$, $g_2(x) = \frac{\|f''\|}{2}x^2 + f(x)$ are convex. By the condition (e) we know for any $n, p > 0$,

$$L_n(g_i, x) - L_{n+p}(g_i, x) \geqslant 0, \quad i = 1, 2.$$

Hence

$$|L_n(f,x)-L_{n+p}(f,x)|\leqslant \frac{\|f''\|}{2}\Big(L_n(e_2,x)-L_{n+p}(e_2,x)\Big), \tag{2.5}$$

$$\|L_n(f)-L_{n+p}(f)\|\leqslant \frac{\|f''\|}{2}\|L_n(e_2)-L_{n+p}(e_2)\|. \tag{2.6}$$

Conditions (d) and (2.6) imply that $(L_n(f))$ is a Cauchy sequence and converges in $C[0,1]$. So there exists an operator L_∞ on $C[0,1]$ such that $\|L_n(f)-L_\infty(f)\|\to 0$ for any $f\in C[0,1]$.

Now let $p\to\infty$ in (2.5), we obtain

$$|L_n(f,x)-L_\infty(f,x)|\leqslant \frac{\|f''\|}{2}\Big(L_n(e_2,x)-L_\infty(e_2,x)\Big)=\frac{\|f''\|}{2}\lambda_n(x). \tag{2.7}$$

Using the equivalence of K-functionals and moduli of smoothness, following the same methods as in the proof of Theorem 5.3 of Chapter 7 in [1], we can get (1.1) from (2.7). Theorem 2 is proved. □

3. Application of Korovkin-type theorem

Let $q\in(0,1]$. For each non-negative integer k, the q-integer $[k]$ and the q-factorial $[k]!$ are defined by

$$[k]=\begin{cases}(1-q^k)\big/(1-q), & q\neq 1,\\ k, & q=1,\end{cases}\qquad [k]!=\begin{cases}[k]\,[k-1]\cdots[1], & k\geqslant 1,\\ 1, & k=0.\end{cases}$$

For the integers $n,\ k,\ n\geqslant k\geqslant 0$, the Gaussian q-binomial coefficients are defined by

$$\begin{bmatrix} n\\ k\end{bmatrix}=\frac{[n]!}{[k]![n-k]!}.$$

In [7], Phillips proposed the following generalization of the Bernstein operators, based on q-integers. For each positive integer n, and $f\in C[0,1]$, we define

$$B_{n,q}(f,x)=\sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right)\begin{bmatrix} n\\ k\end{bmatrix}x^k\prod_{s=0}^{n-k-1}(1-q^s x),\quad 0\leqslant x\leqslant 1,$$

where an empty product denotes 1. When $q=1$, $B_{n,q}(f,x)$ reduces to the well-known Bernstein polynomials $B_n(f,x)$:

$$B_n(f,x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}.$$

Like the classical Bernstein polynomials, the q-Bernstein operators share some good properties. Also a great number of interesting results related to the q-Bernstein operators were obtained (see [2–7]). From [2], we know that q-Bernstein operators satisfy Conditions A, B. But from Theorem 1, Conditions A and B are not sufficient for the convergence of

$(B_{n,q})$. From [4,7], we know that the q-Bernstein operators $B_{n,q}$ reproduce linear functions and satisfy Condition C. It is proved in [7] that $B_{n,q}(e_2, x) = x^2 + x(1-x)/[n]$. Hence

$$B_{n,q}(e_2) \to B_{\infty,q}(e_2), \quad B_{\infty,q}(e_2, x) = x^2 + (1-q)x(1-x).$$

$$|B_{n,q}(e_2, x) - B_{\infty,q}(e_2, x)| = \frac{q^n(1-q)}{1-q^n} x(1-x) \leqslant q^n x(1-x), \quad 0<q<1. \tag{3.1}$$

$$\sup_{0<q<1} |B_{n,q}(e_2, x) - B_{\infty,q}(e_2, x)| = \sup_{0<q<1} \frac{q^n(1-q)}{1-q^n} x(1-x) = \frac{x(1-x)}{n}.$$

Since we know that

$$|B_{n,1}(e_2, x) - B_{\infty,1}(e_2, x)| \leqslant \frac{x(1-x)}{n},$$

we conclude that

$$\sup_{0<q\leqslant 1} |B_{n,q}(e_2, x) - B_{\infty,q}(e_2, x)| \leqslant \frac{x(1-x)}{n}. \tag{3.2}$$

From (3.1), (3.2) and Theorem 2, we obtain that

Theorem 3. Let $0 < q < 1$. Then

$$|B_{n,q}(f, x) - B_{\infty,q}(f, x)| \leqslant c\, \omega_2(f, \sqrt{q^n x(1-x)}\,). \tag{3.3}$$

Furthermore,

$$\sup_{q\in(0,1]} |B_{n,q}(f, x) - B_{\infty,q}(f, x)| \leqslant c\, \omega_2(f, \sqrt{x(1-x)/n}\,). \tag{3.4}$$

where c is the absolute constant.

Remark 1. In [3], it is proved that for each $f \in C[0,1]$, $B_{n,q}(f, x) \to B_{\infty,q}(f, x)$, as $n \to \infty$, uniformly with respect to $x \in [0,1]$ and $q \in [\alpha, 1]$, where $0 < \alpha < 1$, $B_{\infty,1}(f) = f$ and for $0 < q < 1$,

$$B_{\infty,q}(f, x) = \begin{cases} \sum_{k=0}^{\infty} f(1-q^k) \frac{x^k}{(1-q)^k [k]!} \prod_{s=0}^{\infty} (1-q^s x), & 0 \leqslant x < 1, \\ f(1), & x = 1. \end{cases}$$

From (3.4), we conclude that the rates of convergence $\|B_{n,q}(f) - B_{\infty,q}(f)\|$ can be dominated by $c\, \omega_2(f, n^{-1/2})$ uniformly with respect to $q \in (0, 1]$.

Remark 2. In the case $0 < q < 1$, from (3.3) we know that the rates of convergence $\|B_{n,q}(f) - B_{\infty,q}(f)\|$ have the order q^n for the 2 times continuously differentiable function versus $1/n$ for the classical operators.

Remark 3. In [8], Tiberiu Trif introduced the following q-Meyer-König and Zeller operators $M_{n,q}$. For each positive integer n, and $f \in C[0,1]$, we define

$$M_{n,q}(f,x) = \begin{cases} \sum_{k=0}^{\infty} f\left(\frac{[k]}{[n+k]}\right) \begin{bmatrix} n+k \\ k \end{bmatrix} x^k \prod_{j=0}^{n}(1-q^j x), & 0 \leqslant x < 1, \\ f(1), & x = 1. \end{cases}$$

The operators $M_{n,q}$ satisfy Conditions A–C (see [8]). Thus, we conclude that Theorem 3 holds also for $M_{n,q}$.

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